

## FINITE- AND INFINITE-DIMENSIONAL REPRESENTATIONS OF LINEAR SEMISIMPLE GROUPS

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**ABSTRACT.** Every representation in the nonunitary principal series of a noncompact connected real semisimple linear Lie group  $G$  with maximal compact subgroup  $K$  is shown to have a  $K$ -finite cyclic vector. This is used to give a new proof of Harish-Chandra's theorem that every member of the nonunitary principal series has a (finite) composition series. The methods of proof are based on finite-dimensional  $G$ -modules, concerning which some new results are derived. Further related results on infinite-dimensional representations are also obtained.

**1. Introduction.** Let  $G$  be a noncompact connected real semisimple linear Lie group. In this paper, we exploit the relationship between the finite-dimensional  $G$ -modules and the nonunitary principal series to derive some new results and to give simplified proofs of some known results on representations of  $G$ . Perhaps the most striking new result states that every member of the nonunitary principal series has a  $K$ -finite cyclic vector, where  $K$  is the appropriate maximal compact subgroup of  $G$ . One important known result proved here is that every member of the nonunitary principal series has a (finite) composition series. The proof already known (see e.g. [12, Theorem 9.7]), due essentially to Harish-Chandra, uses Harish-Chandra's deep theorem that distribution characters are locally summable functions [7], and related theory. The elementary proof given here is based on a simple algebraic argument of B. Kostant, together with the cyclicity result mentioned above. The cyclicity theorem in turn is based (along with most of the results in this paper) on a sharpening and systematizing of the argument used in [17] to prove the irreducibility of the unitary principal series for complex semisimple Lie groups and for  $SL(2n+1, \mathbb{R})$ . Our methods also enable us to give elementary proofs of Lemme 1 and Lemme 2 of [3].

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The results here on infinite-dimensional modules are generally stated in terms of modules for the complexified universal enveloping algebra  $\mathcal{G}$  of the Lie algebra of  $G$ , but they have obvious analogues for Banach space representations of  $G$ . On the other hand, our restriction to linear groups is essential, since our methods are based on the existence of "enough" finite-dimensional  $G$ -modules. This paper is essentially independent of the authors' previous papers, except for the use of the subquotient theorem (see [5], [12] and [15]) to prove some "finiteness" results concerning  $\mathcal{G}$ -modules. The notation in this paper is cumulative.

We now give a description of the contents.

Let  $G = KAN$  be an Iwasawa decomposition of  $G$ , and let  $M$  be the centralizer of  $A$  in  $K$ . In §2, "extendible" irreducible  $M$ -modules are defined as in [17], and it is shown in two different ways that every irreducible  $M$ -module is extendible (Theorem 2.1), sharpening [17, Theorem 2.1]. Also, a useful explicit description of all the extensions of a given irreducible  $M$ -module is obtained (Theorem 2.6).

In §3, we define the nonunitary principal series as a certain family of  $\mathcal{G}$ -modules, and we derive the (known) description of the finite-dimensional submodules of these modules (Proposition 3.2). This is used, together with a result of S. Helgason on the existence of  $K$ -fixed vectors in finite-dimensional irreducible  $G$ -modules, to prove the following: If two finite-dimensional irreducible  $G$ -modules  $V$  and  $V'$  are extensions (defined in §2) of the same irreducible  $M$ -module, and the highest restricted weight of  $V$  dominates that of  $V'$  in the sense that their difference is dominant, then  $V'$  injects into  $V$  as  $K$ -modules. This generalizes Helgason's theorem, as well as an unpublished result of Kostant. A useful concept of "complete multiplicity" is defined in §3.

In §4, the existence of  $K$ -finite cyclic vectors for the nonunitary principal series modules (Theorem 4.8) is derived, as is the result that every irreducible  $\mathcal{G}$ -module which splits into finite-dimensional irreducible  $K$ -modules in the obvious sense is a subquotient of the tensor product of a nonunitary principal series module with a cyclic  $K$ -fixed vector and a finite-dimensional irreducible  $G$ -module (Theorem 4.10). The proofs are based on a systematization of the methods of [17], the explicit extendibility theorem (Theorem 2.6), and in the case of Theorem 4.10, the subquotient theorem as well. A crucial step is the existence of a large set of nonunitary principal series modules with  $K$ -fixed cyclic vectors. Using deep algebraic methods, Kostant [10] has found all such modules, but we obtain a set large enough for the purposes of this paper from an easier partial result of Helgason [8, p. 129, Lemma 1.2].

In §5, three known finiteness theorems for  $\mathcal{G}$ -modules (including Lemmes 1 and 2 of [3]) are derived from the cyclicity result (Theorem 4.8). The previously known proofs of all these results required Harish-Chandra's deep analysis of

distribution characters, while our proofs are much more elementary and algebraic. As noted above, the simple argument for the finiteness of the composition series for the nonunitary principal series modules (Theorem 5.2) is due to Kostant. In Theorem 5.3, we show using the subquotient theorem and Theorem 5.2 that there are only finitely many inequivalent  $\mathcal{G}$ -modules which split under  $K$  having the same infinitesimal character. Finally, in Theorem 5.5, we indicate among other things that every  $\mathcal{G}$ -module which has an infinitesimal character and which splits into  $K$ -modules with finite multiplicities has a composition series. The results of §5 are known for arbitrary connected noncompact semisimple Lie groups with finite center.

In §6, we prove the existence of finite-dimensional irreducible  $G$ -modules which extend a given irreducible  $M$ -module and which contain a given finite set of irreducible  $K$ -modules with complete multiplicity (Theorem 6.2). The result is used to give an elementary proof of the well-known fact that "almost all" of the nonunitary (and unitary) principal series modules are irreducible (Theorem 6.3). This is a weak form of F. Bruhat's irreducibility theorem [1, p. 193, Théorème 7, 2]. A similar result for complex groups was proved by Harish-Chandra [6, p. 520, Theorem 4].

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**2. Extensions of  $M$ -modules.** Let  $G$  be a noncompact connected real semisimple Lie group with a finite-dimensional faithful representation, and let  $G = KAN$  be an Iwasawa decomposition of  $G$ . Here  $K$  is a maximal compact subgroup of  $G$ ,  $A$  is abelian and  $N$  is nilpotent. Let  $M$  be the centralizer of  $A$  in  $K$  and  $\bar{N} = \theta(N)$ , where  $\theta$  is the Cartan involution of  $G$  associated with  $K$ . If  $H$  is a real Lie group, denote by  $\hat{H}$  the set of equivalence classes of differentiable complex finite-dimensional irreducible  $H$ -modules. It is well known that for all  $\alpha \in \hat{G}$  and  $V \in \alpha$ , the lowest restricted weight space of  $V$  (that is,  $\{v \in V \mid \bar{n} \cdot v = v \text{ for all } \bar{n} \in \bar{N}\}$ ) is invariant and irreducible under  $M$ , giving rise to a well-defined element  $\gamma(\alpha) \in \hat{M}$ .

**Definitions.** We call  $\gamma \in \hat{M}$  *extendible* if  $\gamma = \gamma(\alpha)$  for some  $\alpha \in \hat{G}$ ; such an  $\alpha$  is said to be an *extension* of  $\gamma$  (cf. [17]).

Our first result is a strengthening of the assertion of [17, Theorem 2.1].

**Theorem 2.1.** *Every  $\gamma \in \hat{M}$  is extendible.*

The aim of this section is to prove Theorem 2.1 in two ways. One method proves a more precise statement (Theorem 2.6) which we will need in later sections. The other method is shorter, but for notational reasons it is more convenient to present it second. It consists of filling the gap in the proof of Theorem 2.1

of [17], and noticing that in order to fill the gap one proves Theorem 2.1.

Let  $\mathfrak{g}$ ,  $\mathfrak{k}$ ,  $\mathfrak{a}$ ,  $\mathfrak{n}$  and  $\mathfrak{m}$  be the Lie algebras of  $G$ ,  $K$ ,  $A$ ,  $N$  and  $M$ , respectively, and let  $\mathfrak{t}$  be a maximal abelian subalgebra of  $\mathfrak{m}$ , so that  $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$  is a Cartan subalgebra of  $\mathfrak{g}$ . Denote by  $\mathfrak{g}_C$ ,  $\mathfrak{k}_C$ ,  $\mathfrak{a}_C$ ,  $\mathfrak{n}_C$ ,  $\mathfrak{m}_C$ ,  $\mathfrak{t}_C$  and  $\mathfrak{h}_C$  the respective complexifications. Then  $\mathfrak{t}_C$  is a Cartan subalgebra of  $\mathfrak{m}_C$ , and  $\mathfrak{h}_C = \mathfrak{t}_C + \mathfrak{a}_C$  is a Cartan subalgebra of  $\mathfrak{g}_C$ . Let  $\Delta \subset \mathfrak{h}_C^*$  ( $*$  denotes dual) be the set of roots of  $\mathfrak{g}_C$  with respect to  $\mathfrak{h}_C$ ,  $\Delta^m \subset \mathfrak{t}_C^*$  the set of roots of  $\mathfrak{m}_C$  with respect to  $\mathfrak{t}_C$ , and  $\Sigma \subset \mathfrak{a}_C^*$  the set of restricted roots of  $\mathfrak{g}$  with respect to  $\mathfrak{a}$ , regarded as a set of linear functionals on  $\mathfrak{a}_C$ . We regard  $\mathfrak{t}_C^* \subset \mathfrak{h}_C^*$  and  $\mathfrak{a}_C^* \subset \mathfrak{h}_C^*$  in the natural way resulting from the decomposition  $\mathfrak{h}_C = \mathfrak{t}_C + \mathfrak{a}_C$ . Let  $P: \mathfrak{h}_C^* \rightarrow \mathfrak{t}_C^*$  and  $Q: \mathfrak{h}_C^* \rightarrow \mathfrak{a}_C^*$  denote the restriction maps. All scalar products (sometimes denoted  $(\cdot, \cdot)$ ) in this section will be the forms induced by the Killing form of  $\mathfrak{g}_C$ .

Let  $\Sigma_+ \subset \Sigma$  be the positive system naturally associated with the subgroup  $N$  of  $G$ , and let  $\Delta_+^m \subset \Delta^m$  be an arbitrary positive system for  $\mathfrak{m}$ . Then there is a unique positive system  $\Delta_+ \subset \Delta$  such that  $\Delta_+^m \subset \Delta_+$  and  $Q(\Delta_+ - \Delta_+^m) = \Sigma_+$ . Let  $\Pi \subset \Delta_+$ ,  $\Pi^m \subset \Delta_+^m$  and  $\Pi^a \subset \Sigma_+$  denote the corresponding simple systems. It is known that  $\Pi^m \subset \Pi$  and that (see [16] or [13])  $Q(\Pi - \Pi^m) = \Pi^a$ .

If  $V$  is a  $G$ -module and hence a  $\mathfrak{g}$ -module, we distinguish between the restricted weight spaces of  $V$  (with respect to  $\mathfrak{a}_C$ ) and the weight spaces of  $V$  (with respect to  $\mathfrak{h}_C$ ). The restricted weight space corresponding to the lowest restricted weight agrees with the lowest restricted weight space as defined at the beginning of this section.

Let  $D \subset \mathfrak{h}_C^*$  and  $D^m \subset \mathfrak{t}_C^*$  be the sets of dominant linear forms for  $\mathfrak{g}$  and  $\mathfrak{m}$ , respectively. Specifically,  $D$  is the set of linear forms on  $\mathfrak{h}_C$  which lie in the real span of  $\Delta$  and whose scalar products with the roots in  $\Pi$  are nonnegative;  $D^m$  is the set of linear forms on  $\mathfrak{t}_C$  which are real on  $i\mathfrak{t}$  ( $i = (-1)^{1/2}$ ) and whose values on the root normals  $b_\phi$  ( $\phi \in \Pi^m$ ) are nonnegative (here  $b_\phi \in i\mathfrak{t}$  is that vector in the bracket of the root spaces  $\mathfrak{m}_C^\phi$  and  $\mathfrak{m}_C^{-\phi}$  of  $\mathfrak{m}_C$  such that  $\phi(b_\phi) = 2$ ). Also, let  $D^a \subset \mathfrak{a}_C^*$  be the set of linear forms on  $\mathfrak{a}_C$  which are real on  $\mathfrak{a}$  and whose scalar products with the members of  $\Sigma_+$  are nonnegative.

We may assume that  $G \subset G_C$ , where  $G_C$  is a connected Lie group with Lie algebra  $\mathfrak{g}_C$ . Let  $\mathfrak{p}$  be the orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}$ , so that  $u = \mathfrak{k} + i\mathfrak{p}$  is a compact real form of  $\mathfrak{g}_C$ . Let  $U$ ,  $M_0$ ,  $T$ ,  $A_*$  and  $H_*$  be the connected Lie subgroups of  $G_C$  corresponding to  $u$ ,  $\mathfrak{m}$ ,  $\mathfrak{t}$ ,  $i\mathfrak{a}$  and  $\mathfrak{t} + i\mathfrak{a}$ , respectively. Then  $U$  is a maximal compact subgroup of  $G_C$  with maximal torus  $H_* = TA_*$ ,  $A_*$  is closed and  $M_0$  is the identity component of  $M$ , with maximal torus  $T$ . Writing  $Z = A_* \cap K = A_* \cap M$ , we have that  $M = M_0 Z$  and that  $Z$  is a product of 2-element groups (see [13, pp. 423-424]).

We shall call a linear form  $\mu$  on  $\mathfrak{t}_C$  (resp.,  $\mathfrak{a}_C$ ,  $\mathfrak{h}_C$ ) *integral* if  $\mu$  exponentiates

to a character, also denoted  $\mu$ , of  $T$  (resp.,  $A_*$ ,  $H_*$ ). Often we shall identify integral linear forms with their associated characters. Let  $I^T$  (resp.,  $I^{A*}$ ,  $I^{H*}$ ) be the set of integral linear forms on  $\mathfrak{t}_C$  (resp.,  $\mathfrak{a}_C$ ,  $\mathfrak{h}_C$ ), and let  $I^Z \subset I^{A*}$  be the set of integral linear forms on  $\mathfrak{a}_C$  which are trivial on  $Z$  (when regarded as characters on  $A_*$ ). It is clear that a character on the abelian group  $TZ$  (resp.  $H_*$ ) precisely amounts to a character on  $T$  (i.e., an element of  $I^T$ ) and a character on  $Z$  (resp.,  $A_*$ ) which agree on  $T \cap Z = T \cap A_*$ .

We now describe an indexing of  $\hat{M}$ . Let  $\gamma \in \hat{M}$ ,  $V \in \gamma$ . Then  $Z$  acts as scalars on  $V$  since  $Z$  is central in  $M$ . Since  $M = M_0 Z$ ,  $M_0$  acts irreducibly on  $V$ , and so  $\gamma$  is described by specifying an arbitrary equivalence class of irreducible  $M_0$ -modules and an arbitrary character on  $Z$  which agree on  $M_0 \cap Z$ . On the other hand, a class of irreducible  $M_0$ -modules is determined by its lowest weight, an arbitrary element of  $(-D^m) \cap I^T \subset \mathfrak{t}_C^*$ . But  $M_0 \cap Z \subset T$ , since  $T$  is a maximal abelian subgroup of  $M_0$ . Hence we have the following conclusion:

**Lemma 2.2.** *There is a natural bijection between  $\hat{M}$  and the set of pairs  $(\mu, \chi)$ , where  $\mu \in (-D^m) \cap I^T$ ,  $\chi$  is a character of  $Z$ , and  $\mu$  and  $\chi$  agree on  $T \cap Z$ . To  $\gamma \in \hat{M}$ , we associate the pair  $(\mu(\gamma), \chi(\gamma))$ , where  $\mu(\gamma)$  is the lowest weight of the restriction of  $\gamma$  to  $M_0$ , and  $\chi(\gamma)$  is the restriction of  $\gamma$  to  $Z$ .*

The next lemma relates the lowest weight of  $\alpha \in \hat{G}$  with the data which determine  $\gamma(\alpha)$ .

**Lemma 2.3.** *Let  $\alpha \in \hat{G}$ , and let  $\lambda(\alpha) \in (-D) \cap I^{H*}$  be its lowest weight and  $\chi(\lambda(\alpha))$  the character of  $Z$  obtained by restricting  $\lambda(\alpha)$  to  $Z$ . Then*

$$P(\lambda(\alpha)) = \mu(\gamma(\alpha)) \quad \text{and} \quad \chi(\lambda(\alpha)) = \chi(\gamma(\alpha)).$$

*Equivalently,  $\alpha \in \hat{G}$  is an extension of  $\gamma \in \hat{M}$  if and only if*

$$P(\lambda(\alpha)) = \mu(\gamma) \quad \text{and} \quad \chi(\lambda(\alpha)) = \chi(\gamma).$$

**Proof.** Let  $V \in \alpha$ , and let  $Y$  be the lowest restricted weight space of  $V$  and  $v \in V$  a nonzero lowest weight vector of  $V$ . Then  $v$  is a weight vector for  $\mathfrak{t}_C$  with weight  $P(\lambda(\alpha))$ . Since the root spaces  $\mathfrak{g}^{-\phi}$  ( $\phi \in \Delta_+ - \Delta_+^m$ ) annihilate  $v$ ,  $v \in Y$ , and since the root spaces  $\mathfrak{g}^{-\phi}$  ( $\phi \in \Delta_+^m$ ) annihilate  $v$ ,  $v$  is a lowest weight vector for the  $M_0$ -module  $Y$ . Thus  $v$  is a weight vector for  $\mathfrak{t}_C$  with weight  $\mu(\gamma(\alpha))$ , proving the first assertion. The assertion  $\chi(\lambda(\alpha)) = \chi(\gamma(\alpha))$  is clear, and the last statement now follows immediately from Lemma 2.2. Q.E.D.

The next two lemmas implement the procedures suggested naturally by Lemma 2.3:

**Lemma 2.4.** *Let  $\mu \in -D^m$ . Then  $P^{-1}(\mu) \cap (-D) \subset \mathfrak{h}_C^*$  is a translate of  $-D^a$ . Specifically,*

$$P^{-1}(\mu) \cap (-D) = \mu + \nu_\mu - D^a,$$

where  $\nu_\mu \in \mathfrak{a}_C^*$  is determined by the condition

$$(\nu_\mu, \phi) = - \max_{\psi \in Q^{-1}(\phi) \cap \Pi} (\mu, \psi)$$

for all  $\phi \in \Pi^a$ .

**Proof.** Clearly,  $P^{-1}(\mu) = \{\mu + \nu \mid \nu \in \mathfrak{a}_C^*\}$ , so that

$$P^{-1}(\mu) \cap (-D) = \{\mu + \nu \mid \nu \in \mathfrak{a}_C^*, \nu \text{ real on } \mathfrak{a} \text{ and}$$

$$(\mu + \nu, \psi) \leq 0 \text{ for all } \psi \in \Pi\}$$

But if  $\nu \in \mathfrak{a}_C^*$  and  $\psi \in \Pi^m$ , then

$$(\mu + \nu, \psi) = (\mu, \psi) = \frac{1}{2}(\psi, \psi)\mu(b_\psi) \leq 0$$

since  $\mu \in -D^m$ . Hence

$$P^{-1}(\mu) \cap (-D) = \{\mu + \nu \mid \nu \in \mathfrak{a}_C^*, \nu \text{ real on } \mathfrak{a} \text{ and}$$

$$(\nu, \psi) \leq -(\mu, \psi) \text{ for all } \psi \in \Pi - \Pi^m\}$$

and this implies the lemma. Q.E.D.

**Lemma 2.5.** *Let  $\sigma$  be a character of  $TZ$ , with  $\mu \in I^T$  the corresponding character of  $T$  and  $\chi$  the corresponding character of  $Z$ . Then  $\chi$  extends to a character of  $A_*$ , and if  $\nu \in I^{A_*}$  is any extension, then the set of all extensions is the set  $\nu + I^Z$ . Moreover,  $\sigma$  extends to a character of  $H_*$ , and if  $\lambda \in I^{H_*}$  is any extension, then  $P(\lambda) = \mu$ , and the set of all extensions is the set  $\lambda + I^Z$ . Also,  $I^Z$  is the set of extensions to  $H_*$  of the trivial character of  $TZ$ .*

**Proof.** The group  $Z'$  of all elements of order 2 of  $A_*$  is a product of 2-element groups, and so  $\chi$  extends to a character of  $Z'$  and hence to a character of  $A_*$ . The rest is clear. Q.E.D.

We now have the main theorem:

**Theorem 2.6.** *Let  $\gamma \in \hat{M}$ . Then  $\gamma$  is extendible, and the set of lowest weights of extensions of  $\gamma$  is exactly the nonempty set*

$$(\mu(\gamma) + \nu_{\mu(\gamma)} - D^a) \cap (\lambda + I^Z),$$

where  $\lambda \in I^{H*}$  is any extension to  $H_*$  of the character of  $TZ$  defined by  $\mu(\gamma)$  and  $\chi(\gamma)$ , and  $\nu_{\mu(\gamma)} \in \alpha_C^*$  is defined in Lemma 2.4.

**Proof.** Let  $\alpha \in \hat{G}$ , with  $\lambda(\alpha)$  its lowest weight. In view of Lemma 2.3,  $\alpha$  extends  $\gamma$  if and only if (1)  $P(\lambda(\alpha)) = \mu(\gamma)$  and (2)  $\chi(\lambda(\alpha)) = \chi(\gamma)$ . By Lemma 2.4, (1) and the condition that  $\lambda(\alpha) \in -D$  assert that  $\lambda(\alpha) \in \mu(\gamma) + \nu_{\mu(\gamma)} - D^a$ . By Lemma 2.5, (1) and (2) and the integrality of  $\lambda(\alpha)$  are equivalent to the condition that  $\lambda(\alpha) \in \lambda + I^Z$ , where  $\lambda \in I^{H*}$  exists as stated in the theorem. The non-emptiness of the intersection in the theorem is clear. Q.E.D.

**Remark 2.7.** If  $\gamma$  is the trivial element of  $\hat{M}$ , we may of course take  $\nu_{\mu(\gamma)} = \lambda = 0$  in Theorem 2.6, so that the lowest weights of the extensions of  $\gamma$  constitute the set  $(-D^a) \cap I^Z$ . Let us denote this set by  $S$ , so that the members of  $S$  are the lowest weights of the irreducible  $G$ -modules whose lowest restricted weight spaces are one-dimensional and trivial under  $M$ . By a theorem of S. Helgason [8, p. 79, Corollary 3.8] these are exactly the (finite-dimensional irreducible) spherical  $G$ -modules, that is, those which contain a nonzero  $K$ -fixed vector. Hence  $S$  is the set of lowest weights of spherical  $G$ -modules.

**Remark 2.8.** One might conjecture that in the notation of Theorem 2.6,  $\mu(\gamma) + \nu_{\mu(\gamma)}$  itself is an extension to  $H_*$  of the character of  $TZ$  defined by  $\mu(\gamma)$  and  $\chi(\gamma)$ , so that the lowest weights of the extensions of  $\gamma$  constitute the set  $\mu(\gamma) + \nu_{\mu(\gamma)} + S$ . This is true if  $G$  is complex but it fails in many cases, for example in the case  $G = \text{SL}(n, \mathbb{R})$ . However, it would be interesting to determine whether the set of lowest weights of extensions of  $\gamma$  is a translate of  $S$ . This happens frequently, and would set up a natural bijection between the set of equivalence classes of spherical irreducible  $G$ -modules and the set of equivalence classes of irreducible  $G$ -modules which extend any given element of  $\hat{M}$ .

We now give the alternate proof of Theorem 2.1 adjusting the proof of Theorem 2.1 in [17].

Let  $E$  be the space of continuous functions on  $M$  spanned by the matrix elements of the  $\gamma(\alpha)$  for  $\alpha \in \hat{G}$ . In view of the Stone-Weierstrass theorem, Theorem 2.1 follows from (1), (2) and (3) below:

(1)  $E$  is an algebra under multiplication. In fact, if  $\alpha_1, \alpha_2 \in \hat{G}$  then  $\gamma(\alpha_1) \otimes \gamma(\alpha_2)$  is a submodule of  $\Sigma_{m_\delta} \gamma(\delta)$  where  $\alpha_1 \otimes \alpha_2 = \Sigma m_\delta \delta$  ( $\delta \in \hat{G}$ ).

(2)  $E$  is closed under complex conjugation. Indeed, if  $\alpha \in \hat{G}$  let  $\bar{\alpha}$  be the class gotten by taking complex conjugates of the matrices of  $\alpha$  relative to a basis in a realization of  $\alpha$ . Then  $\gamma(\bar{\alpha})$  has matrix entries which are complex conjugates of matrix entries of  $\gamma(\alpha)$ .

(3)  $E$  separates the points of  $M$ . In fact, let  $m \in M$  and suppose that  $f(m) = f(e)$  for all  $f \in E$ . (Here  $e$  is the identity element of  $G$ .) To show that  $m = e$ , note first  $f(m_1 m m_1^{-1}) = f(e)$  for all  $f \in E$ ,  $m_1 \in M$ . Hence we may assume that  $m \in TZ \subset TA_* = H_*$ , a maximal torus of  $U$ . It is thus sufficient to show that any  $m \in H_*$  which acts as the identity on  $V^{\bar{N}}$  for all  $V \in \alpha$ ,  $\alpha \in \hat{G}$  must be the identity element of  $H_*$ . But from the proof of Lemma 2.3 above, we see that the lowest weight space of  $V$  is contained in  $V^{\bar{N}}$ . Hence it is sufficient to show that the characters of  $H_*$  which lie in  $(-D) \cap I^{H*}$  separate the points of  $H_*$ . This will follow if we prove that the functionals in the additive subgroup of  $\mathfrak{h}_C^*$  generated by  $(-D) \cap I^{H*}$  separate the points of  $H_*$  (when regarded as characters on  $H_*$ ). But the set of all characters in  $I^{H*}$  separates the points of  $H_*$ , so that it remains only to show that  $I^{H*}$  is the subgroup of  $\mathfrak{h}_C^*$  generated by  $(-D) \cap I^{H*}$ . The following lemma, which will also be referred to in the proof of Theorems 4.8 and 4.9, asserts slightly more than what we need here:

**Lemma 2.9.** *Every element  $\lambda$  of  $\mathfrak{h}_C^*$  which lies in the real span of  $\Delta$  is of the form  $\lambda' - \lambda''$ , where  $\lambda' \in D$  and  $\lambda'' \in D \cap I^{H*}$ . If  $\lambda \in I^{H*}$ , then  $\lambda' \in D \cap I^{H*}$ .*

**Proof.** The Killing form of  $\mathfrak{g}_C$  is nonsingular on  $\mathfrak{h}_C$  and so induces an isomorphism between  $\mathfrak{h}_C$  and  $\mathfrak{h}_C^*$ . For all  $\psi \in \Pi$ , let  $x_\psi$  be the image of  $\psi$  in  $\mathfrak{h}_C$  under this isomorphism. Then  $\xi \in \mathfrak{h}_C^*$  is in  $D$  if and only if  $\xi(x_\psi) \geq 0$  for all  $\psi \in \Pi$ . Suppose that  $\lambda \in \mathfrak{h}_C^*$  is in the real span of  $\Pi$ . If  $\lambda(x_\psi) \geq 0$  for all  $\psi \in \Pi$ , then we may take  $\lambda' = \lambda$  and  $\lambda'' = 0$ , and the lemma is proved. If  $\lambda(x_\psi) < 0$  for some  $\psi \in \Pi$ , define  $\lambda_1, \lambda_2 \in \mathfrak{h}_C^*$  by the conditions

$$\lambda_1(x_\psi) = \lambda(x_\psi) - \min_{\phi \in \Pi} \lambda(x_\phi) \quad \text{and} \quad \lambda_2(x_\psi) = - \min_{\phi \in \Pi} \lambda(x_\phi)$$

for all  $\psi \in \Pi$ . Then  $\lambda_1, \lambda_2 \in D$  and  $\lambda = \lambda_1 - \lambda_2$ . Clearly there exists  $\lambda'' \in I^{H*}$  such that  $\lambda''(x_\psi) \geq \lambda_2(x_\psi)$  for all  $\psi \in \Pi$ . Then  $\lambda'' \in D \cap I^{H*}$ ,  $\lambda' = \lambda_1 + (\lambda'' - \lambda_2) \in D$ , and  $\lambda = \lambda' - \lambda''$ . This proves the first assertion, and the last assertion is immediate. Q.E.D.

**3. The nonunitary principal series and finite-dimensional  $G$ -modules.** In this section, we define the standard family of infinite-dimensional  $\mathfrak{g}$ -modules called the nonunitary principal series, determine their finite-dimensional submodules, and use the result to derive a result on multiplicities of finite-dimensional irreducible  $K$ -modules in finite-dimensional irreducible  $G$ -modules. To the previous notation, we add a few definitions taken from [12]. The notation used in this paper to describe the nonunitary principal series is slightly different from that used in [12], for technical reasons of convenience.



Let  $\mathcal{G}$  be the universal enveloping algebra of  $\mathfrak{g}_{\mathbb{C}}$ . A  $\mathcal{G}$ -module  $V$  which is also a  $K$ -module is called a  $(\mathcal{G}, K)$ -module if

$$k \cdot (x \cdot v) = (k \cdot x) \cdot (k \cdot v)$$

for all  $k \in K$ ,  $x \in \mathcal{G}$  and  $v \in V$ . (Here  $K$  acts on  $\mathcal{G}$  according to the natural extension of the adjoint action of  $K$  on  $\mathfrak{g}$ .) If in addition, for all  $v \in V$ ,  $K \cdot v$  spans a finite-dimensional space on which  $K$  acts differentiably and  $\mathfrak{k} \subset \mathcal{G}$  acts according to the differential of the action of  $K$ , then  $V$  is said to be a *compatible*  $(\mathcal{G}, K)$ -module.  $(\mathcal{G}, K)$ -module maps and equivalence of  $(\mathcal{G}, K)$ -modules are defined in the obvious ways. The tensor product of (compatible)  $(\mathcal{G}, K)$ -modules is a (compatible)  $(\mathcal{G}, K)$ -module in the natural way.

Let  $\mathcal{K}$  be the universal enveloping algebra of  $\mathfrak{k}_{\mathbb{C}}$ , regarded as canonically embedded in  $\mathcal{G}$ . The notions of (compatible)  $(\mathcal{G}, G)$ - and  $(\mathcal{K}, K)$ -module and related concepts are defined as above.

The equivalence class of a module  $V$  is denoted by  $[V]$ . For every  $K$ -module  $V$  and  $\beta \in \hat{K}$ , let  $m([V], \beta)$  denote the multiplicity with which members of  $\beta$  occur in  $V$ , and similarly for other groups. We regard  $G$ -modules as  $K$ -modules by restriction, so that for example if  $\alpha \in \hat{G}$  and  $\beta \in \hat{K}$ ,  $m(\alpha, \beta)$  is the multiplicity with which members of  $\beta$  occur in any member of  $\alpha$ ; similarly for other pairs of groups, such as  $K, M$ . An element  $v$  of a  $K$ -module  $V$  is called *K-finite* if  $K \cdot v$  spans a finite-dimensional subspace of  $V$  on which  $K$  acts differentiably; similarly for *G-finite*. An element  $v$  of a  $\mathcal{K}$ -module  $V$  is called *K-finite* if  $\mathcal{K} \cdot v$  is finite-dimensional; similarly for *G-finite*.

We now turn to the definition of the family of compatible  $(\mathcal{G}, K)$ -modules which we call the nonunitary principal series. These modules are the dense modules consisting of the  $K$ -finite vectors in the corresponding Hilbert space completions—the usual nonunitary principal series of representations of  $G$ . In this paper, we treat only the infinitesimal modules. The reader is referred to [18, §3] for the definition of the Hilbert space modules, and to [4] for their relationship with the infinitesimal modules.

Let  $\log: A \rightarrow \mathfrak{a}$  be the inverse of the diffeomorphism  $\exp: \mathfrak{a} \rightarrow A$ .

Fix  $\gamma \in \hat{M}$ ,  $\nu \in \mathfrak{a}_{\mathbb{C}}^*$  and  $Y \in \gamma$ , and define an  $MAN$ -module structure on  $Y$  as follows:

$$man \cdot y = e^{\nu(\log a)}_m \cdot y$$

for all  $m \in M$ ,  $a \in A$ ,  $n \in N$  and  $y \in Y$ . Call this  $MAN$ -module  $\tilde{Y}$ . Let  $X^{(Y, \nu)}$  be the space of all  $C^\infty$  functions  $f: G \rightarrow \tilde{Y}$  such that  $f(gp) = p^{-1} \cdot f(g)$  for all  $g \in G$  and  $p \in MAN$ . Then  $X^{(Y, \nu)}$  is a  $G$ -module under the action given by

$(g \cdot f)(b) = f(g^{-1}b)$  for all  $f \in X^{(Y, \nu)}$  and  $g, b \in G$ . Also,  $X^{(Y, \nu)}$  is a  $\mathfrak{g}$ -module, and hence a  $\mathfrak{g}_C$ -module and a  $\mathcal{G}$ -module, by means of the action

$$(x \cdot f)(g) = \frac{d}{dt} f((\exp - tx)g) \Big|_{t=0}$$

for all  $f \in X^{(Y, \nu)}$ ,  $x \in \mathfrak{g}$  and  $g \in G$ . This makes  $X^{(Y, \nu)}$  a  $(\mathcal{G}, G)$ -module.

Let  $X^{Y, \nu}$  be the subspace of  $K$ -finite vectors of  $X^{(Y, \nu)}$ . Then  $X^{Y, \nu}$  is  $\mathcal{G}$ -invariant, and is a compatible  $(\mathcal{G}, K)$ -module. As  $\gamma$  ranges through  $\hat{M}$  and  $\nu$  ranges through  $\mathfrak{a}_C^*$ , the compatible  $(\mathcal{G}, K)$ -modules  $X^{Y, \nu}$  ( $Y \in \gamma$ ) constitute the *non-unitary principal series* of  $G$ . It is clear that the classes  $[X^{Y, \nu}]$  and  $[X^{(Y, \nu)}]$  depend only on  $\gamma$  and  $\nu$  (and not on  $Y$ ).

In order to describe the multiplicities of  $K$ -modules in the modules  $X^{Y, \nu}$ , we need the following additional construction:

Denote by  $U^{(Y)}$  the space of all  $C^\infty$  functions  $f: K \rightarrow Y$  such that  $f(km) = m^{-1} \cdot f(k)$  for all  $k \in K$  and  $m \in M$ . Then  $U^{(Y)}$  is a  $K$ -module and in fact a  $(K, K)$ -module under the actions given by  $(k \cdot f)(l) = f(k^{-1}l)$  for all  $f \in U^{(Y)}$  and  $k, l \in K$ , and

$$(x \cdot f)(k) = \frac{d}{dt} ((\exp - tx)k) \Big|_{t=0}$$

for all  $f \in U^{(Y)}$ ,  $x \in \mathfrak{k}$  and  $k \in K$ .

Let  $U^Y$  be the subspace of  $K$ -finite vectors in  $U^{(Y)}$ , so that  $U^{(Y)}$  is  $K$ -invariant, and is a compatible  $(K, K)$ -module.

The following simple but important (known) result tells us the multiplicities of  $K$ -modules in the nonunitary principal series modules:

**Proposition 3.1.** *The restriction map to  $K$   $R^{Y, \nu}: X^{Y, \nu} \rightarrow U^Y$  is a  $(K, K)$ -module isomorphism with inverse*

$$((R^{Y, \nu})^{-1}(f))(kan) = e^{-\nu(\log a)} f(k)$$

for all  $f \in U^Y$ ,  $k \in K$ ,  $a \in A$  and  $n \in N$ . Also, for all  $\beta \in \hat{K}$ ,  $m([U^Y], \beta) = m(\beta, \gamma)$ , so that  $m([X^{Y, \nu}], \beta) = m(\beta, \gamma)$ .

**Proof.** The second part of the proposition follows immediately from the Frobenius reciprocity theorem for the pair  $(K, M)$  of compact groups, together with the first part of the proposition, which in turn is straightforward. Q.E.D.

Let  $\delta_{Y, \nu}: X^{(Y, \nu)} \rightarrow \tilde{Y}$  be the evaluation map at the identity element of  $G$ . It is trivial to check that  $\delta_{Y, \nu}$  is an  $MAN$ -module map, and that  $\delta_{Y, \nu}$  is *effective* in the following sense: For all nonzero  $f \in X^{(Y, \nu)}$ ,  $\delta_{Y, \nu}(G \cdot f) \neq 0$ .

We now describe the extent to which the finite-dimensional irreducible  $G$ -modules occur in nonunitary principal series modules.

For any  $\alpha \in \hat{G}$ , let  $\nu(\alpha) \in \alpha_C^*$  be the lowest restricted weight of any member of  $\alpha$ . For all  $\gamma \in \hat{M}$ ,  $\nu \in \alpha_C^*$  and  $Y \in \gamma$ , define  $X_0^{Y,\nu}$  to be the compatible  $(\mathcal{G}, G)$ -module of  $G$ -finite vectors in  $X^{(Y,\nu)}$ . Then clearly  $X_0^{Y,\nu} \subset X^{Y,\nu}$ , and  $X_0^{Y,\nu}$  is exactly the  $\mathcal{G}$ -module of  $\mathcal{G}$ -finite vectors in  $X^{Y,\nu}$ .

The following useful result is known (cf. for example [18, Theorem 4.2]), but we include a simple proof for completeness.

**Proposition 3.2.** *If  $\gamma \in \hat{M}$ ,  $\nu \in \alpha_C^*$  and  $Y \in \gamma$ , then  $X_0^{Y,\nu}$  is either 0 or  $G$ - (and hence  $\mathfrak{g}$ -) irreducible. More precisely  $X^{(Y,\nu)}$  contains a nonzero finite-dimensional  $G$ -submodule if and only if  $\gamma = \gamma(\alpha)$  and  $\nu = \nu(\alpha)$  for some  $\alpha \in \hat{G}$ , and in this case,  $X_0^{Y,\nu}$  is an irreducible  $G$ -module in the class  $\alpha$ . Equivalently, given  $\alpha \in \hat{G}$ , there is exactly one pair  $\gamma, \nu$  ( $\gamma \in \hat{M}$ ,  $\nu \in \alpha_C^*$ ) such that if  $Y \in \gamma$ ,  $\alpha$  occurs in  $X^{(Y,\nu)}$ , namely, the pair  $\gamma = \gamma(\alpha)$ ,  $\nu = \nu(\alpha)$ , and  $\alpha$  occurs with multiplicity exactly one in  $X^{(Y,\nu)}$  in this case.*

**Proof.** Let  $\alpha \in \hat{G}$ , and  $Z \in \gamma(\alpha)$ . We first show that  $\alpha$  occurs in  $X^{(Z,\nu(\alpha))}$ . Let  $V \in \alpha$ ,  $Y$  the lowest restricted weight space of  $V$ , and  $p: V \rightarrow Y$  the projection with respect to the restricted weight space decomposition of  $V$ . Define a map  $\theta$  from  $V$  into the space of  $Y$ -valued functions on  $G$  by  $\theta(v)(g) = p(g^{-1} \cdot v)$  for all  $v \in V$  and  $g \in G$ . It is straightforward to check that  $\theta$  is a  $G$ -module injection of  $V$  into  $X^{(Y,\nu(\alpha))}$ , showing that  $\alpha$  occurs in  $X^{(Y,\nu(\alpha))} \simeq X^{(Z,\nu(\alpha))}$ .

Now let  $\gamma \in \hat{M}$ ,  $\nu \in \alpha_C^*$  and  $Y \in \gamma$ , and let  $W$  be any finite-dimensional  $G$ -submodule of  $X^{(Y,\nu)}$ . Then the restriction of  $\delta_{Y,\nu}$  (see above) to  $W$  is an  $MAN$ -module map into  $\tilde{Y}$ . In particular, since  $\tilde{Y}$  is trivial as an  $N$ -module,  $\delta_{Y,\nu}$  annihilates  $n \cdot W$  (regarding  $W$  as a  $\mathfrak{g}$ -module). Let  $U$  be the  $MA$ -submodule of  $\bar{N}$ -fixed vectors of  $W$  (see the beginning of §2 for the definition of  $\bar{N}$ ). Then  $\delta_{Y,\nu}$  restricts to an  $MA$ -module map  $\eta$  from  $U$  into  $Y$ .

We assert that  $\eta$  is injective. Indeed, let  $V$  be the  $G$ -submodule of  $W$  generated by  $\text{Ker } \eta$ . Since  $\text{Ker } \eta$  is an  $MA$ -submodule of  $U$ , it is easy to see that  $V = n \cdot V + \text{Ker } \eta$ . Thus  $\delta_{Y,\nu}$  vanishes on  $V$ . Hence  $V = 0$  since  $\delta_{Y,\nu}$  is effective (see above), and this proves the assertion since  $\text{Ker } \eta \subset V$ .

Thus either  $W = 0$  or  $W \in \alpha$ , where  $\alpha \in \hat{G}$  is such that  $\gamma = \gamma(\alpha)$  and  $\nu = \nu(\alpha)$ . In particular, if  $X^{(Y,\nu)}$  contains a nonzero finite-dimensional  $G$ -submodule, then  $\gamma = \gamma(\alpha)$  and  $\nu = \nu(\alpha)$  for some  $\alpha \in \hat{G}$ , and  $X_0^{Y,\nu} \in \alpha$ . Combining this with the first paragraph of the proof, we have the proposition. Q.E.D.

As an immediate consequence of Proposition 3.1 and (the first paragraph of the proof of) Proposition 3.2, we have the well-known result:

**Corollary 3.3.** *For all  $\alpha \in \hat{G}$  and  $\beta \in \hat{K}$ ,  $m(\alpha, \beta) \leq m(\beta, \gamma(\alpha))$ .*

**Remark 3.4.** Given  $\alpha \in \hat{G}$ , it is an interesting question to determine for which  $\beta \in \hat{K}$  the inequality in Corollary 3.3 becomes an equality. One important result of this type is Helgason's theorem mentioned in Remark 2.7 above; this asserts the equality when  $\beta$  and  $\gamma(\alpha)$  are trivial. Other such results are the existence of "minimal types" for representations of complex semisimple Lie groups [14, p. 394, Corollary 1] and the analogous result for real rank one groups announced in [11, Theorem 5]. In this paper, we shall give other results (Corollary 3.9 and Theorem 6.2) asserting the equality in certain cases. Theorem 6.2 provides a simple proof of the well-known result that the unitary and nonunitary principal series are "almost everywhere" irreducible (Theorem 6.3).

We introduce the following terminology for the concept discussed in the last remark:

**Definition.** Let  $\alpha \in \hat{G}$  and  $\beta \in \hat{K}$ . We say that  $\alpha$  contains  $\beta$  with complete multiplicity or that  $\beta$  occurs in  $\alpha$  with complete multiplicity if  $m(\alpha, \beta) = m(\beta, \gamma(\alpha))$ .

Let  $\gamma \in \hat{M}$ ,  $\nu, \nu' \in \alpha_C^*$  and  $Y \in \gamma$ . Regard  $\mathbb{C}$  as the trivial  $M$ -module. Now  $X^{(Y, \nu)}$  consists of functions from  $G$  into  $Y$ , and  $X^{(C, \nu')}$  consists of functions from  $G$  into  $\mathbb{C}$ . Hence pointwise multiplication of functions on  $G$  induces a linear map  $\omega$  from  $X^{(C, \nu')} \otimes X^{(Y, \nu)}$  into the space of  $Y$ -valued functions on  $G$ . The proof of the following lemma is straightforward:

**Lemma 3.5.** With the above notation, the image of  $\omega$  is contained in  $X^{(Y, \nu + \nu')}$ , and

$$\omega: X^{(C, \nu')} \otimes X^{(Y, \nu)} \rightarrow X^{(Y, \nu + \nu')}$$

is a  $(\mathcal{G}, G)$ -module map, where the domain of  $\omega$  is the tensor product of  $(\mathcal{G}, G)$ -modules. Moreover,  $\omega$  restricts to a  $(\mathcal{G}, G)$ -module map

$$\omega: X_0^{C, \nu'} \otimes X_0^{Y, \nu} \rightarrow X_0^{Y, \nu + \nu'}$$

and to a  $(\mathcal{G}, K)$ -module map

$$\omega: X^{C, \nu'} \otimes X^{Y, \nu} \rightarrow X^{Y, \nu + \nu'}.$$

The following key lemma generalizes the basic idea of Lemma 2.3 of [17]:

**Lemma 3.6.** Define  $1_{\nu'} \in X^{(C, \nu')}$  as follows:

$$1_{\nu'}(kan) = e^{-\nu'(\log a)},$$

where  $k \in K$ ,  $a \in A$  and  $n \in N$ . Then  $1_{\nu'}$  is the unique element of  $X^{(C, \nu')}$  which is the constant 1 on  $K$ , and is the unique (up to scalar)  $K$ -fixed vector in

$X^{C,\nu'}$ . The restriction of  $\omega$  (see Lemma 3.5) to  $1_{\nu'} \otimes X^{Y,\nu}$  induces a  $(K, K)$ -module isomorphism  $\iota: X^{Y,\nu} \rightarrow X^{Y,\nu+\nu'}$ , and  $\iota = (R^{Y,\nu+\nu'})^{-1} \circ R^{Y,\nu}$  (see Proposition 3.1). That is, identifying  $X^{Y,\nu}$  and  $X^{Y,\nu+\nu'}$  with their restrictions to  $K$  according to Proposition 3.1,  $\iota$  is the identity map on  $U^Y$ .

**Proof.** It is clear that  $1_{\nu'} \in X^{(C,\nu')}$  and that  $1_{\nu'}$  is the unique element which is 1 on  $K$ . Since the  $K$ -fixed vectors in  $X^{(C,\nu')}$  are exactly the elements which are constant on  $K$ ,  $1_{\nu'} \in X^{C,\nu'}$ , and  $1_{\nu'}$  is the unique (up to scalar)  $K$ -fixed vector. The last assertions are clear. Q.E.D.

We define a partial vector space ordering  $\leq$  on the real span of  $\Sigma$  in  $\mathfrak{a}_C^*$  by the condition that  $D^{\mathfrak{a}}$  consists of the positive elements (and 0). The following main result of this section relates this partial ordering to certain  $K$ -module inclusions:

**Theorem 3.7.** Let  $\alpha, \alpha' \in \hat{G}$ ,  $V \in \alpha$ ,  $V' \in \alpha'$  and suppose that  $\gamma(\alpha) = \gamma(\alpha')$ . If  $\nu(\alpha') \leq \nu(\alpha)$ , then  $m(\alpha, \beta) \leq m(\alpha', \beta)$  for all  $\beta \in \hat{K}$ , or, equivalently, there is a  $K$ -module injection of  $V$  into  $V'$ . More precisely, the map  $\iota$  of Lemma 3.6 induces a  $(K, K)$ -module injection from  $X_0^{Y,\nu(\alpha)}$  into  $X_0^{Y,\nu(\alpha')}$  (where  $Y \in \gamma(\alpha)$ ).

**Proof.** From Lemma 3.5, we have the  $G$ -module map

$$\omega: X_0^{C,\nu(\alpha')-\nu(\alpha)} \otimes X_0^{Y,\nu(\alpha)} \rightarrow X_0^{Y,\nu(\alpha')},$$

where  $Y \in \gamma(\alpha)$ . Now  $\nu(\alpha') - \nu(\alpha) \in -D^{\mathfrak{a}}$  by hypothesis, and  $\nu(\alpha') - \nu(\alpha) \in I^Z$  by Theorem 2.6. Hence  $\nu(\alpha') - \nu(\alpha) \in S$ , in the notation of Remark 2.7. In view of Remark 2.7, there exists  $\alpha'' \in \hat{G}$  such that  $\gamma(\alpha'')$  is trivial and  $\nu(\alpha'') = \nu(\alpha') - \nu(\alpha)$ , and Proposition 3.2 implies that  $X_0^{C,\nu(\alpha')-\nu(\alpha)} \in \alpha''$ . Hence by Helgason's theorem (see Remark 2.7),  $X_0^{C,\nu(\alpha')-\nu(\alpha)}$  has a nonzero  $K$ -fixed vector. Thus  $1_{\nu(\alpha')-\nu(\alpha)} \in X_0^{C,\nu(\alpha')-\nu(\alpha)}$  (see Lemma 3.6), and so by Lemma 3.6, the restriction of  $\omega$  (see above) to  $1_{\nu(\alpha')-\nu(\alpha)} \otimes X_0^{Y,\nu(\alpha)}$  induces a  $K$ -module injection of  $X_0^{Y,\nu(\alpha)}$  into  $X_0^{Y,\nu(\alpha')}$ . But  $X_0^{Y,\nu(\alpha)} \in \alpha$  and  $X_0^{Y,\nu(\alpha')} \in \alpha'$  by Proposition 3.2, proving the theorem. Q.E.D.

**Remark 3.8.** In the special case in which  $\mathfrak{g}$  is split over the reals, the first assertion of Theorem 3.7 was proved by B. Kostant [unpublished] by different methods.

Corollary 3.3 and Theorem 3.7 imply the following:

**Corollary 3.9.** Let  $\alpha_0 \in \hat{G}$  and  $\beta \in \hat{K}$ , and assume that  $\alpha_0$  contains  $\beta$  with complete multiplicity. Then any  $\alpha \in \hat{G}$  such that  $\gamma(\alpha) = \gamma(\alpha_0)$  and  $\nu(\alpha) \leq \nu(\alpha_0)$  also contains  $\beta$  with complete multiplicity.

**Remark 3.10.** It is interesting to note that Corollary 3.9 implies Helgason's

theorem (see Remark 2.7), by choosing  $\alpha_0$  and  $\beta$  to be the trivial classes in  $\hat{G}$  and  $\hat{K}$ , respectively. But Helgason's theorem is of course used in the proof of Corollary 3.9.

**4. Cyclic vectors from spherical cyclic vectors.** In this section, we systematize the key arguments used in [17, Theorem 3.2] to construct cyclic vectors for nonunitary principal series modules. The results here will be used to prove the finiteness theorems of §5 and the irreducibility theorem of §6. The last result in this section uses the "subquotient" theorem for linear groups  $G$ , proved by Harish-Chandra (for linear groups) in [5] and simplified in [12] and [15].

Let  $\gamma \in \hat{M}$  and  $Y \in \gamma$ . As in §3, regard  $\mathbb{C}$  as the trivial  $M$ -module, and let  $\zeta$  be the linear map from  $U^{(\mathbb{C})} \otimes U^{(Y)}$  into the space of functions from  $K$  into  $Y$  induced by pointwise multiplication. The proof of the following analogue of Lemma 3.5 is straightforward:

**Lemma 4.1.** *The image of  $\zeta$  is contained in  $U^{(Y)}$ , and*

$$\zeta: U^{(\mathbb{C})} \otimes U^{(Y)} \rightarrow U^{(Y)}$$

*is a  $(K, K)$ -module map which restricts to a  $(K, K)$ -module map*

$$\zeta: U^{\mathbb{C}} \otimes U^Y \rightarrow Y^Y.$$

The following key result is implicit in the proof of Theorem 3.2 of [17]:

**Proposition 4.2.** *Let  $\gamma \in \hat{M}$ ,  $Y \in \gamma$  and  $V$  any nonzero  $K$ -submodule of  $U^{(Y)}$ . Then  $\zeta(U^{(\mathbb{C})} \otimes V) = U^{(Y)}$ , and if  $V \subset U^Y$ , then  $\zeta(U^{\mathbb{C}} \otimes V) = U^Y$ .*

**Proof.** Let  $e$  be the identity element of  $G$ . Choose a nonzero element  $f \in V$ , and assume that  $k \in K$  is such that  $f(k) \neq 0$ . Then  $(k^{-1} \cdot f)(e) \neq 0$ , so that we may assume that  $f \in V$  satisfies  $f(e) \neq 0$ . But then  $(M \cdot f)(e) = M \cdot (f(e))$  spans  $Y$  since  $Y$  is  $M$ -irreducible, and so we may choose  $f_1, \dots, f_n \in V$  (of the form  $m \cdot f$  for certain  $m \in M$ ) such that  $f_1(e), \dots, f_n(e)$  constitute a basis of  $Y$ .

Let  $\pi: K \rightarrow K/M$  be the natural projection mapping, and choose a local cross-section for  $M$  in  $K$ , that is, a  $C^\infty$  function  $\xi$  from an open neighborhood  $U$  of  $\{M\} \in K/M$  into  $K$  such that  $\pi\xi$  is the identity on  $U$  and such that the map  $\rho: U \times M \rightarrow \pi^{-1}(U)$  given by  $\rho(u, m) = \xi(u)m$  ( $u \in U$ ,  $m \in M$ ) is a  $C^\infty$  diffeomorphism.

By shrinking  $U$  if necessary, we may assume that  $f_1(k), \dots, f_n(k)$  form a basis of  $Y$  for all  $k \in \xi(U)$ . Choose  $k_1, \dots, k_p \in K$  such that  $k_1\pi^{-1}(U), \dots, k_p\pi^{-1}(U)$  cover  $K$ , and for all  $i = 1, \dots, n$  and  $j = 1, \dots, p$ , let  $f_{ij} = k_j \cdot f_i$ . Then for all  $j = 1, \dots, p$  and  $k \in k_j\xi(U)$ ,  $f_{1j}(k), \dots, f_{nj}(k)$  form a basis of  $Y$ . (The last statement also holds for all  $k \in k_j\pi^{-1}(U)$ , since the  $f_{ij} \in U^{(Y)}$ .)

Fix a  $C^\infty$  partition of unity  $\phi_1, \dots, \phi_p$  subordinate to the covering  $k_1 \cdot U, \dots, k_p \cdot U$  of  $K/M$  (where the left action of  $K$  on  $K/M$  is denoted with a dot), and let  $\psi_j = \phi_j \circ \pi$  ( $j = 1, \dots, p$ ). Then each  $\psi_j \in U^{(C)}$ , and  $\psi_1, \dots, \psi_p$  form a  $C^\infty$  partition of unity subordinate to the covering  $k_1\pi^{-1}(U), \dots, k_p\pi^{-1}(U)$  of  $K$ .

Let  $b \in U^{(Y)}$ . For each  $j = 1, \dots, p$ , it is clear that there are  $C^\infty$  complex-valued functions  $b_{1j}, \dots, b_{nj}$  on  $k_j\pi^{-1}(U)$ , constant on the left  $M$ -cosets, such that

$$b|_{k_j\pi^{-1}(U)} = \sum_{i=1}^n b_{ij}(f_{ij}|_{k_j\pi^{-1}(U)}).$$

Then  $\psi_j b_{1j}, \dots, \psi_j b_{nj}$  are well-defined functions in  $U^{(C)}$ , and

$$b = \sum_{i,j} (\psi_j b_{ij}) f_{ij} \in \zeta(U^{(C)} \otimes V),$$

proving the first assertion of the proposition.

To prove the second assertion, we note first that if  $V \subset U^Y$ , then  $\zeta(U^C \otimes V) \subset U^Y$ . Now  $U^C$  is uniformly dense in  $U^{(C)}$ , and so  $\zeta(U^C \otimes V)$  is uniformly dense in  $\zeta(U^{(C)} \otimes V) = U^{(Y)}$ , and hence in  $U^Y$ . But the only  $K$ -invariant subspace of  $U^Y$  uniformly dense in  $U^Y$  is  $U^Y$  itself, and this proves the desired result. Q.E.D.

The next result is an easy consequence of Proposition 4.2:

**Theorem 4.3.** *Let  $\gamma \in \hat{M}$ ,  $\nu, \nu' \in \alpha_C^*$  and  $Y \in \gamma$ . Let  $W$  be any nonzero  $\mathcal{G}$ -submodule of  $X^{Y, \nu}$ . Then  $\omega$  (see Lemma 3.5) induces a  $\mathcal{G}$ -module surjection from  $X^{C, \nu'} \otimes W$  onto  $X^{Y, \nu + \nu'}$ .*

**Proof.** We first note that

$$R^{Y, \nu + \nu'}(\omega(X^{C, \nu'} \otimes W)) = \zeta(U^C \otimes R^{Y, \nu}(W))$$

(see Proposition 3.1 and Lemma 4.1). Since  $R^{Y, \nu}(W)$  is a nonzero  $\mathcal{K}$ -submodule, and hence  $K$ -submodule, of  $U^Y$ , Proposition 4.2 implies that

$$R^{Y, \nu + \nu'}(\omega(X^{C, \nu'} \otimes W)) = U^Y.$$

Thus

$$\omega(X^{C, \nu'} \otimes W) = X^{Y, \nu + \nu'}. \quad \text{Q.E.D.}$$

We can now establish the following general result (also implicit in the proof of Theorem 3.2 of [17]) on the construction of cyclic vectors:

**Theorem 4.4.** *Let  $\mathfrak{g}$  be any Lie subalgebra of  $\mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{k} + \mathfrak{g}$  (not necessarily direct). Let  $\gamma \in \hat{M}$ ,  $\nu, \nu' \in \alpha_C^*$  and  $Y \in \gamma$ , and let  $f_0 \in X^{C, \nu'}$  and*

$f_1 \in X^{Y, \nu}$  span one-dimensional spaces which are  $\mathfrak{k}$ -invariant and  $\mathfrak{s}$ -invariant, respectively. Then  $f_0 \otimes f_1$  is a  $\mathcal{G}$ -cyclic vector for the  $\mathcal{G}$ -module  $\mathcal{G} \cdot f_0 \otimes \mathcal{G} \cdot f_1$ . If in addition  $f_0$  is a  $\mathcal{G}$ -cyclic vector for the  $\mathcal{G}$ -module  $X^{C, \nu'}$ , then  $\omega(f_0 \otimes f_1)$  (see Lemma 3.5) is a  $\mathcal{G}$ -cyclic vector for the  $\mathcal{G}$ -module  $X^{Y, \nu + \nu'}$ .

**Proof.** In view of Theorem 4.3, it is sufficient to prove that  $f_0 \otimes f_1$  is  $\mathcal{G}$ -cyclic for  $\mathcal{G} \cdot f_0 \otimes \mathcal{G} \cdot f_1$ .

Let  $\mathcal{S} \subset \mathcal{G}$  be the complexified universal enveloping algebra of  $\mathfrak{s}$ . Since  $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$  we have  $\mathcal{G} = \mathcal{K}\mathcal{S} = \mathcal{S}\mathcal{K}$ .

For all  $n = 0, 1, 2, \dots$ , let  $\mathcal{K}_n \subset \mathcal{K}$  (resp.,  $\mathcal{S}_n \subset \mathcal{S}$ ) be the span of the products of at most  $n$  elements of  $\mathfrak{k}$  (resp.,  $\mathfrak{s}$ ). It is easy to prove by induction on  $n$  that

$$\mathcal{K}_n \cdot (f_0 \otimes f_1) = f_0 \otimes \mathcal{K}_n \cdot f_1,$$

so that

$$\mathcal{K} \cdot (f_0 \otimes f_1) = f_0 \otimes \mathcal{K} \cdot f_1 = f_0 \otimes \mathcal{G} \cdot f_1$$

since  $\mathcal{G} = \mathcal{K}\mathcal{S}$ , and so

$$\mathcal{G} \cdot (f_0 \otimes f_1) = \mathcal{S} \cdot (f_0 \otimes \mathcal{G} \cdot f_1),$$

since  $\mathcal{G} = \mathcal{S}\mathcal{K}$ .

Similarly, for all  $n = 0, 1, 2, \dots$ , we have

$$\mathcal{S}_n \cdot (f_0 \otimes \mathcal{G} \cdot f_1) = \mathcal{S}_n \cdot f_0 \otimes \mathcal{G} \cdot f_1,$$

so that

$$\mathcal{S} \cdot (f_0 \otimes \mathcal{G} \cdot f_1) = \mathcal{S} \cdot f_0 \otimes \mathcal{G} \cdot f_1 = \mathcal{G} \cdot f_0 \otimes \mathcal{G} \cdot f_1,$$

since  $\mathcal{G} = \mathcal{S}\mathcal{K}$ . Thus

$$\mathcal{G} \cdot (f_0 \otimes f_1) = \mathcal{G} \cdot f_0 \otimes \mathcal{G} \cdot f_1. \quad \text{Q.E.D.}$$

**Corollary 4.5.** Let  $\gamma \in \hat{M}$ ,  $\nu \in \mathfrak{a}^*$  and  $Y \in \gamma$ . Also, let  $\alpha \in \hat{G}$  be any extension of  $\gamma$  and  $V \in \alpha$ . Suppose that  $X_{\mathcal{C}, \nu}^{C, \nu}$  has a  $\mathcal{G}$ -cyclic  $K$ -fixed vector. Then  $X^{Y, \nu + \nu(\alpha)}$  and  $X^{C, \nu} \otimes V$  are cyclic  $\mathcal{G}$ -modules.

**Proof.** The corollary follows immediately from Theorem 4.4 by choosing  $\mathfrak{s} = \alpha + \mathfrak{n}$  and  $f_1$  any nonzero highest restricted weight vector in  $X_0^{Y, \nu(\alpha)}$ . Q.E.D.



**Remark 4.6.** Theorem 4.4 allows considerable flexibility in the choice of the cyclic vectors for  $X^{Y, \nu + \nu(\alpha)}$  and  $X^{C, \nu} \otimes V$  in Corollary 4.5. We can choose  $\mathfrak{g}$  to be any Iwasawa complement of  $\mathfrak{k}$  in  $\mathfrak{g}$ , and  $f_1$  be to any nonzero highest weight vector in  $X_0^{Y, \nu(\alpha)}$  with respect to that Iwasawa complement.

In view of Corollary 4.5, the problem of finding cyclic vectors for nonunitary principal series modules is reduced to the problem of finding  $K$ -fixed cyclic vectors for the spherical nonunitary principal series. Let  $\alpha_R^*$  be the real dual of  $\alpha$ , and define  $\rho \in \alpha_C^*$  by  $\rho(x) = \frac{1}{2} \text{tr}(\text{ad } x | \mathfrak{n}_C)$  for all  $x \in \alpha_C$ . In B. Kostant [10] it is shown that if  $\nu \in \rho + D^{\mathfrak{a}} + i \alpha_R^*$  (see §2 for the definition of  $D^{\mathfrak{a}}$ ) then  $1_\nu$  is a cyclic vector for  $X^{C, \nu}$ . This result is quite hard and for the purposes of this paper the easier proof of S. Helgason [8, p. 129, Lemma 1.2] that  $1_\nu$  is cyclic if  $\nu \in \rho + D_0^{\mathfrak{a}} + i \alpha_R^*$ , where  $D_0^{\mathfrak{a}}$  is the set of real linear forms on  $\mathfrak{a}$  whose scalar products with the members of  $\Sigma_+$  are positive, will suffice. We state the result in the following form:

**Proposition 4.7.** *There exists  $\nu_0 \in \alpha_R^*$  so that if  $\nu \in \nu_0 + D^{\mathfrak{a}} + i \alpha_R^*$ , then  $1_\nu$  is a cyclic vector for  $X^{C, \nu}$ .*

The following two theorems are proved simultaneously:

**Theorem 4.8.** *Every member of the nonunitary principal series of  $G$  is cyclic, and hence finitely generated, as a  $\mathcal{G}$ -module.*

**Theorem 4.9.** *Every member of the nonunitary principal series of  $G$  is a  $\mathcal{G}$ -module quotient of the tensor product of a spherical nonunitary principal series module whose  $K$ -fixed vector is  $\mathcal{G}$ -cyclic with a finite-dimensional irreducible  $G$ -module (regarded as a  $\mathcal{G}$ -module).*

**Proof.** Let  $\gamma \in \hat{M}$  and  $\nu \in \alpha_C^*$ . In view of Theorem 4.3 and Corollary 4.5, it is sufficient to find  $\nu' \in \alpha_C^*$  and  $\alpha \in \hat{G}$  such that the  $K$ -fixed vector in  $X^{C, \nu'}$  is  $\mathcal{G}$ -cyclic,  $\alpha$  is an extension of  $\gamma$ , and  $\nu = \nu' + \nu(\alpha)$ . By Theorem 2.6, the set of  $\nu(\alpha)$  such that  $\alpha \in \hat{G}$  is an extension of  $\gamma$  is of the form  $(\nu_{\mu(\gamma)} - D^{\mathfrak{a}}) \cap (\lambda_1 + I^Z)$ , where  $\lambda_1$  is the restriction of  $\lambda$  to  $\alpha_C$ , in the notation of Theorem 2.6. By Proposition 4.7, the set of  $\nu' \in \alpha_C^*$  such that the  $K$ -fixed vector in  $X^{C, \nu'}$  is cyclic contains a set of the form  $\nu_0 + D^{\mathfrak{a}} + i \alpha_R^*$ . Write  $\nu = \nu_1 + i \nu_2$ , where  $\nu_1, \nu_2 \in \alpha_R^*$ . An argument similar to the proof of the first assertion of Lemma 2.9 shows that  $\nu_1$  lies in the set  $(\nu_0 + D^{\mathfrak{a}}) + ((\nu_{\mu(\gamma)} - D^{\mathfrak{a}}) \cap (\lambda_1 + I^Z))$ . Hence  $\nu$  is of the form  $\nu' + \nu(\alpha)$  as desired. Q.E.D.

**Theorem 4.10.** *Every  $\mathcal{G}$ -irreducible compatible  $(\mathcal{G}, K)$ -module is equivalent to a subquotient (i.e., a quotient of submodules) of the tensor product of a spherical nonunitary principal series module whose  $K$ -fixed vector is  $\mathcal{G}$ -cyclic with a*

*finite-dimensional irreducible  $G$ -module (regarded as a  $\mathcal{G}$ -module).*

**Proof.** In view of the subquotient theorem ([5, p. 63, Theorem 4], [12, Theorem 8.8] or [15]), every  $\mathcal{G}$ -irreducible compatible  $(\mathcal{G}, K)$ -module is equivalent to a subquotient of a nonunitary principal series module. The result now follows from Theorem 4.9. Q.E.D.

**5. Finiteness theorems.** Here we use the above results to prove a series of "finiteness" theorems (in addition to Theorems 4.8, 4.9 and 4.10) on various types of  $\mathcal{G}$ -modules. The last two (Theorems 5.3 and 5.5) require the subquotient theorem for linear groups  $G$  (see [5], [12] and [15]). Theorems 5.2, 5.5 and the ingredients of 5.3 are already known (see [12] and [3]), with the hard part of the known proofs due to Harish-Chandra. The argument used to derive Theorem 5.2 from Theorem 4.8 is due to Kostant. Theorems 5.3 and 5.5 include Lemmes 1 and 2 of [3].

In order to prove that every member of the nonunitary principal series has a (finite) composition series, we need the following standard lemma, whose proof we omit:

**Lemma 5.1.** *Let  $\gamma \in \hat{M}$ ,  $\nu \in \mathfrak{a}_C^*$  and  $Y \in \gamma$ , and regard  $Y^*$  as the  $M$ -module contragredient to  $Y$ . Denote by  $\langle \cdot, \cdot \rangle$  the pairing between  $Y^*$  and  $Y$ . Define  $\rho \in \mathfrak{a}_C^*$  by  $\rho(x) = \frac{1}{2} \text{tr}(\text{ad } x | \mathfrak{n}_C)$  for all  $x \in \mathfrak{a}_C$ . Then there is a natural  $\mathcal{G}$ -module map*

$$r: X^{Y^*, 2\rho-\nu} \otimes X^{Y, \nu} \rightarrow X^{C, 2\rho}$$

*given by*

$$f_1 \otimes f_2 \mapsto f_3$$

*where  $f_3(g) = \langle f_1(g), f_2(g) \rangle$  for all  $g \in G$ . Moreover, the bilinear pairing*

$$B: X^{Y^*, 2\rho-\nu} \times X^{Y, \nu} \rightarrow \mathbb{C}$$

*given by*

$$f_1, f_2 \mapsto \int_K (\kappa f_1 \otimes f_2 | K) dk,$$

*where  $dk$  is Haar measure on  $K$ , is  $\mathcal{G}$ -invariant. Finally, every  $K$ -invariant subspace of  $X^{Y, \nu}$  or of  $X^{Y^*, 2\rho-\nu}$  is its own double annihilator with respect to  $B$ .*

**Theorem 5.2.** *Every member of the nonunitary principal series has a (finite) composition series as a  $\mathcal{G}$ -module.*

**Proof (Kostant).** Let  $\gamma \in \hat{M}$ ,  $\nu \in \mathfrak{a}_C^*$  and  $Y \in \gamma$ . By Theorem 4.8,  $X^{Y, \nu}$  is

finitely generated as a  $\mathcal{G}$ -module. Hence  $X^{Y,\nu}$  satisfies the ascending chain condition for submodules, since  $\mathcal{G}$  is a Noetherian ring (see [9, p. 166]). The same is true of  $X^{Y^*, 2\rho-\nu}$ . But by the last assertion of Lemma 5.1, the ascending chain condition for  $X^{Y^*, 2\rho-\nu}$  is equivalent to the descending chain condition for  $X^{Y,\nu}$ . Since  $X^{Y,\nu}$  now satisfies both chain conditions, it has a composition series. Q.E.D.

Let  $\mathcal{Z}$  be the center of  $\mathcal{G}$ . If  $V$  is a  $\mathcal{G}$ -module on which  $\mathcal{Z}$  acts as scalars, the resulting homomorphism from  $\mathcal{Z}$  into  $\mathbb{C}$  is called the *infinitesimal character* of  $V$ . A well-known result of J. Dixmier asserts that every irreducible  $\mathcal{G}$ -module has an infinitesimal character (see for example [12, Lemma 2.2]). It is also known that inequivalent irreducible  $\mathcal{G}$ -modules can have the same infinitesimal character (see [2, p. 19–09]). This is true even if the  $\mathcal{G}$ -modules are assumed to be compatible  $(\mathcal{G}, K)$ -modules. The next theorem illuminates this situation, but first we need some notation.

Let  $\mathfrak{n}_{\mathcal{G}} \subset \mathfrak{g}_{\mathcal{C}}$  be the sum of the positive root spaces of  $\mathfrak{g}_{\mathcal{C}}$  with respect to  $\mathfrak{h}_{\mathcal{C}}$ , and let  $\mathcal{H} \subset \mathcal{G}$  be the universal enveloping algebra of  $\mathfrak{h}_{\mathcal{C}}$ . It is easy to see that  $\mathcal{Z} \subset \mathcal{H} + \mathfrak{n}_{\mathcal{G}}$ , and that this sum is direct. Let  $\chi^{\mathfrak{g}}: \mathcal{Z} \rightarrow \mathcal{H}$  be the corresponding projection map. Then it is straightforward to check that  $\chi^{\mathfrak{g}}$  is an algebra homomorphism. Now  $\mathcal{H}$  is naturally isomorphic to the algebra of complex-valued polynomial functions on  $\mathfrak{h}_{\mathcal{C}}^*$ , so that for all  $\lambda \in \mathfrak{h}_{\mathcal{C}}^*$ , the evaluation homomorphism  $\pi_{\lambda}: \mathcal{H} \rightarrow \mathbb{C}$  is well defined. Define the homomorphism  $\chi_{\lambda}: \mathcal{Z} \rightarrow \mathbb{C}$  by  $\chi_{\lambda} = \pi_{\lambda} \circ \chi^{\mathfrak{g}}$ . Let  $\gamma \in \hat{M}$ ,  $\nu \in \mathfrak{a}_{\mathcal{C}}^*$  and  $Y \in \gamma$ , and let  $\mu(\gamma) \in \mathfrak{t}_{\mathcal{C}}^*$  be the lowest weight of the  $\mathfrak{m}_{\mathcal{C}}$ -module  $Y$ , as in Lemma 2.2. Regarding  $\mu(\gamma)$  and  $\nu$  as linear functionals on  $\mathfrak{h}_{\mathcal{C}}^*$  by defining them to be 0 on  $\mathfrak{a}_{\mathcal{C}}$  and  $\mathfrak{t}_{\mathcal{C}}$ , respectively, we get an element  $\mu(\gamma) + \nu$  of  $\mathfrak{h}_{\mathcal{C}}^*$ . Denote  $\chi_{\mu(\gamma)+\nu}$  by  $\chi_{\gamma,\nu}$ .

Let  $W$  be the Weyl group of  $\mathfrak{g}_{\mathcal{C}}$  with respect to  $\mathfrak{h}_{\mathcal{C}}$ , regarded as acting on  $\mathfrak{h}_{\mathcal{C}}^*$ , and let  $\rho' \in \mathfrak{h}_{\mathcal{C}}^*$  be half the sum of the positive roots of  $\mathfrak{g}_{\mathcal{C}}$  with respect to  $\mathfrak{h}_{\mathcal{C}}$ . A well-known theorem of Harish-Chandra states that every homomorphism from  $\mathcal{Z}$  into  $\mathbb{C}$  is of the form  $\chi_{\lambda}$  for some  $\lambda \in \mathfrak{h}_{\mathcal{C}}^*$ , and that  $\chi_{\lambda} = \chi_{\lambda'}$  ( $\lambda, \lambda' \in \mathfrak{h}_{\mathcal{C}}^*$ ) if and only if  $\lambda - \rho'$  and  $\lambda' - \rho'$  are in the same  $W$ -orbit of  $\mathfrak{h}_{\mathcal{C}}^*$  (see [2, §§18, 19]). For every homomorphism  $\chi: \mathcal{Z} \rightarrow \mathbb{C}$ , let  $S(\chi)$  be the set of all pairs  $\gamma, \nu$  ( $\gamma \in \hat{M}$ ,  $\nu \in \mathfrak{a}_{\mathcal{C}}^*$ ) such that if  $\lambda \in \mathfrak{h}_{\mathcal{C}}^*$  is chosen so that  $\chi = \chi_{\lambda}$ , then  $\lambda - \rho'$  and  $\mu(\gamma) + \nu - \rho'$  are in the same  $W$ -orbit. Then clearly  $S(\chi)$  is a finite set, and is the set of all pairs  $\gamma, \nu$  such that  $\chi = \chi_{\gamma,\nu}$ .

**Theorem 5.3.** *Let  $\chi: \mathcal{Z} \rightarrow \mathbb{C}$  be a homomorphism. Then there exists a  $\mathcal{G}$ -irreducible compatible  $(\mathcal{G}, K)$ -module with the infinitesimal character  $\chi$  if and only if  $S(\chi)$  is nonempty. There are only finitely many inequivalent  $\mathcal{G}$ -irreducible compatible  $(\mathcal{G}, K)$ -modules with the infinitesimal character  $\chi$ , and these modules are precisely (up to equivalence) the composition quotients of the modules  $X^{Y,\nu}$  where  $Y \in \gamma$ ,  $\gamma \in \hat{M}$ ,  $\nu \in \mathfrak{a}_{\mathcal{C}}^*$ , and the pair  $\gamma, \nu$  ranges through  $S(\chi)$ .*

**Proof.** By the subquotient theorem ([4, p. 63, Theorem 4], [12, Theorem 8.8] or [15]), every  $\mathcal{G}$ -irreducible compatible  $(\mathcal{G}, K)$ -module is equivalent to a subquotient of some member of the nonunitary principal series  $X^{Y, \nu}$  ( $Y \in \gamma$ ,  $\gamma \in \hat{M}$ ,  $\nu \in \mathfrak{a}_C^*$ ). By Theorem 5.2,  $X^{Y, \nu}$  has a composition series, and so the irreducible subquotients of  $X^{Y, \nu}$  are exactly (up to equivalence) its composition quotients, and there are only finitely many of these. On the other hand, it is well known that  $X^{Y, \nu}$ , and hence any of its composition quotients, has an infinitesimal character equal to  $\chi_{\gamma, \nu}$  (see e.g. [12, Proposition 8.9]). The theorem now follows easily from the comments preceding the theorem. Q.E.D.

**Remark 5.4.** Theorems 5.2 and 5.3 hold for noncompact connected real semisimple Lie groups with finite center as well as for linear groups (see [12, Theorem 9.7 and Corollary 9.15]). The proofs can proceed as follows: First one uses Harish-Chandra's deep results to prove that there are only finitely many infinitesimally inequivalent irreducible Hilbert space representations of  $G$  with a given infinitesimal character (see the first part of the proof of Lemme 1 of [3]). Then one proves the assertion of Theorem 5.2, using for example the argument given in the proof of Theorem 9.16 of [12]. Finally, the assertion of Theorem 5.3 is proved as in this paper, using the subquotient theorem for not necessarily linear groups with finite center (see [12, Theorem 8.8] or [15]).

Theorem 5.3 implies the following additional finiteness result, whose statement and proof for linear groups  $G$  are the same as those of [12, Theorem 9.16] for groups  $G$  with finite center:

**Theorem 5.5.** *Let  $V$  be a compatible  $(\mathcal{G}, K)$ -module with an infinitesimal character. Then the following conditions are equivalent:*

- (1) *For all  $\beta \in \hat{K}$ ,  $m([V], \beta) < \infty$ .*
- (2) *There is a constant  $C$  such that, for all  $\beta \in \hat{K}$ ,  $m([V], \beta) \leq C d(\beta)$  (where  $d(\beta)$  denotes the dimension of any module in the class  $\beta$ ).*
- (3)  *$V$  is a finitely generated  $\mathcal{G}$ -module.*
- (4)  *$V$  is a Noetherian  $\mathcal{G}$ -module.*
- (5)  *$V$  has a composition series as a  $\mathcal{G}$ -module.*

**6. Complete multiplicity and an irreducibility result.** In this section we show that for all  $\gamma \in \hat{M}$  and  $\beta \in \hat{K}$  there exists an extension  $\alpha \in \hat{G}$  of  $\gamma$  which contains  $\beta$  with complete multiplicity (Theorem 6.2). We use this result to give a simple proof of a weak form of F. Bruhat's irreducibility theorem [1, p. 193, Théorème 7, 2] for the principal series (see Theorem 6.3). A similar result was proved by Harish-Chandra for complex semisimple Lie groups [6, p. 520, Theorem 4].

**Lemma 6.1.** *Let  $V \subset U^C$  be an arbitrary finite-dimensional subspace. Then*

there exists an extension  $\alpha \in \hat{G}$  of the trivial  $M$ -module such that

$$V \subset R^{C, \nu(\alpha)}(X_0^{C, \nu(\alpha)})$$

(see Proposition 3.1).

**Proof.** We recall that  $\rho \in \alpha_C^*$  is defined by  $\rho(x) = \frac{1}{2} \text{tr}(\text{ad } x | \mathfrak{n}_C)$  for all  $x \in \mathfrak{a}_C$ . Then it is easy to see that  $-2\rho \in S$ , in the notation of Remark 2.7. We shall show that the elements of  $X_0^{C, -2\rho}$  separate the points of  $K/M$ . For all  $g = kan \in G$  ( $k \in K$ ,  $a \in A$ ,  $n \in N$ ), define  $H(g) = \log a$ . Then  $1_{-2\rho}$  (see Lemma 3.6) is given by  $1_{-2\rho}(g) = e^{2\rho(H(g))}$  for all  $g \in G$ , and  $1_{-2\rho} \in X_0^{C, -2\rho}$ . Let  $k \in K$ , and suppose that  $f(k) = f(e)$  for all  $f \in X_0^{C, -2\rho}$ . We must show that  $k \in M$ . But for all  $g \in G$ ,  $(g \cdot 1_{-2\rho})(k) = (g \cdot 1_{-2\rho})(e)$ , that is,  $e^{2\rho(H(g^{-1}k))} = e^{2\rho(H(g^{-1}))}$ . Hence  $e^{-2\rho(H(g^{-1}k))} = e^{-2\rho(H(g^{-1}))}$ , so that  $(g \cdot 1_{2\rho})(k) = (g \cdot 1_{2\rho})(e)$ . Differentiating, we get  $(x \cdot 1_{2\rho})(k) = (x \cdot 1_{2\rho})(e)$  for all  $x \in \mathcal{G}$ . But in the discussion preceding Proposition 4.7, it was noted that we may take  $\nu_0 = 2\rho$  in Proposition 4.7. Thus  $f(k) = f(e)$  for all  $f \in X^{C, 2\rho}$  and hence for all  $f \in U^C$  (see Proposition 3.1). This shows that  $k \in M$ , proving our assertion.

Since  $X_0^{C, -2\rho}$  is closed under complex conjugation, the algebra  $C$  of continuous functions on  $K/M$  generated by  $R^{C, -2\rho}(X_0^{C, -2\rho})$  is uniformly dense in the space of all continuous functions, by the last paragraph and the Stone-Weierstrass theorem. Hence  $C = U^C$ . But for all nonnegative integers  $k$  and  $l$ ,

$$\omega(X_0^{C, -2k\rho} \otimes X_0^{C, -2l\rho}) = X_0^{C, -2(k+l)\rho}$$

(see Proposition 3.2 and Lemma 3.5). Thus since  $\omega$  is just pointwise multiplication, and since

$$R^{C, -2k\rho}(X_0^{C, -2k\rho}) \subset R^{C, -2l\rho}(X_0^{C, -2l\rho})$$

if  $k \leq l$ , we have

$$C = \bigcup_{k=0}^{\infty} R^{C, -2k\rho}(X_0^{C, -2k\rho}),$$

and this proves the proposition. Q.E.D.

The following result generalizes Lemma 6.1:

**Theorem 6.2.** Let  $\gamma \in \hat{M}$ ,  $Y \in \gamma$ , and let  $S \subset U^Y$  be an arbitrary finite-dimensional subspace. Then there exists an extension  $\alpha \in \hat{G}$  of  $\gamma$  such that  $S \subset R^{Y, \nu(\alpha)}(X_0^{Y, \nu(\alpha)})$ . In particular, for all  $\beta_1, \dots, \beta_n \in \hat{K}$ , there exists an extension  $\alpha \in \hat{G}$  of  $\gamma$  such that  $\beta_1, \dots, \beta_n$  occur in  $\alpha$  with complete multiplicity.

**Proof.** Let  $\alpha' \in \hat{G}$  be any extension of  $\gamma$ . By Proposition 4.2,

$$\zeta(U^C \otimes R^{Y, \nu(\alpha')} (X_0^{Y, \nu(\alpha')})) = U^Y,$$

and so there exists a finite-dimensional subspace  $V \subset U^C$  such that

$$\zeta(V \otimes R^{Y, \nu(\alpha')} (X_0^{Y, \nu(\alpha')})) \supset S.$$

But by Lemma 6.1, there is an extension  $\alpha'' \in \hat{G}$  of the trivial  $M$ -module such that

$$\begin{aligned} S &\subset \zeta(R^{C, \nu(\alpha'')} (X_0^{C, \nu(\alpha'')})) \otimes R^{Y, \nu(\alpha')} (X_0^{Y, \nu(\alpha')}) \\ &= R^{Y, \nu(\alpha') + \nu(\alpha'')} (\omega(X_0^{C, \nu(\alpha'')})) \otimes X_0^{Y, \nu(\alpha')} \\ &\subset R^{Y, \nu(\alpha') + \nu(\alpha'')} (X_0^{Y, \nu(\alpha') + \nu(\alpha'')}) \end{aligned}$$

(by Lemma 3.5), and this proves the first assertion. The remaining assertion follows by applying the first assertion to the sum of the  $\beta_1, \dots, \beta_n$ -primary subspaces of  $U^Y$  to construct an extension  $\alpha$  of  $\gamma$  such that  $m(\alpha, \beta_i) = m([U^Y], \beta_i) = m(\beta_i, \gamma)$  for each  $i = 1, \dots, n$ . Q.E.D.

We now prove a weak irreducibility theorem for the unitary and nonunitary principal series modules. In our notation, the modules  $X^{Y, \rho + i\nu}$  for  $Y \in \gamma \in \hat{M}$  and  $\nu \in \mathfrak{a}_R^*$  (the real dual of  $\mathfrak{a}$ ) correspond to the unitary principal series. Let  $\mathcal{S}$  be the set of all  $\nu \in \mathfrak{a}_C^*$  such that for all  $\gamma \in \hat{M}$  and  $Y \in \gamma$ ,  $X^{Y, \nu}$  is  $\mathcal{G}$ -irreducible.

**Theorem 6.3.**  $\mathcal{S}$  (resp.,  $\mathcal{S} \cap (\rho + i\mathfrak{a}_R^*)$ ) is nonempty. Moreover, there is a countable family  $\mathcal{F}$  of nonzero complex-valued polynomial functions on  $\mathfrak{a}_C^*$  such that if  $\mathcal{T}$  is the complement of the union of the algebraic hypersurfaces defined by the members of  $\mathcal{F}$ , then  $\mathcal{T} \subset \mathcal{S}$ . In particular,  $\mathcal{S}$  (resp.,  $\mathcal{S} \cap (\rho + i\mathfrak{a}_R^*)$ ) is dense in  $\mathfrak{a}_C^*$  (resp.,  $\rho + i\mathfrak{a}_R^*$ ) and its complement has measure zero.

**Proof.** Let  $\gamma \in \hat{M}$ ,  $Y \in \gamma$ . Let  $\beta_1, \beta_2, \beta_3, \dots$  be all the members of  $\hat{K}$ . For each  $n = 1, 2, \dots$ , let

$$P_n: U^Y \rightarrow \sum_{i=1}^n U_{\beta_i}^Y$$

be the projection map with respect to the  $K$ -primary decomposition of  $U^Y$ , where the subscript  $\beta_i$  denotes the  $\beta_i$ -primary component. Also, let

$$E_n = \text{End} \left( \sum_{i=1}^n U_{\beta_i}^Y \right).$$

For each  $\nu \in \mathfrak{a}_C^*$ , let  $L_n^\nu \subset E_n$  be the space

$$P_n \circ R^{Y, \nu} \circ \pi_{Y, \nu}(\mathcal{G}) \circ (R^{Y, \nu})^{-1} \circ P_n,$$

where  $\pi_{Y,\nu}$  is the action of  $\mathcal{G}$  on  $X^{Y,\nu}$ . Clearly  $X^{Y,\nu}$  is irreducible if  $L_n^\nu = E_n$  for all  $n$ .

Let  $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \dots$  be the standard filtration of  $\mathcal{G}$ . For each  $\nu \in \alpha_{\mathbb{C}}^*$ ,  $n = 1, 2, \dots$  and  $k = 1, 2, \dots$ , define

$$F_{n,k}^\nu: \mathcal{G}_k \rightarrow E_n$$

by

$$x \mapsto P_n \circ R^{Y,\nu} \circ \pi_{Y,\nu}(x) \circ (R^{Y,\nu})^{-1} \circ P_n.$$

Then it is easy to see that the map

$$F_{n,k}: \alpha_{\mathbb{C}}^* \rightarrow \text{Hom}(\mathcal{G}_k, E_n)$$

given by

$$\nu \mapsto F_{n,k}^\nu$$

is a polynomial map of degree at most  $k$ .

For every  $n > 0$ , there exists  $\nu_n \in \alpha_{\mathbb{C}}^*$  such that  $L_n^{\nu_n} = E_n$ . Indeed, it is sufficient to choose an extension  $\alpha \in \hat{G}$  of  $\gamma$  such that

$$\sum_{i=1}^n U_{\beta_i}^Y \subset R^{Y,\nu(\alpha)}(X_0^{Y,\nu(\alpha)})$$

by Theorem 6.2; we may take  $\nu_n = \nu(\alpha)$ . Then there exists  $k_n > 0$  such that

$F_{n,k_n}^{\nu_n}: \mathcal{G}_{k_n} \rightarrow E_n$  is surjective. Choose a subspace  $S_n \subset \mathcal{G}_{k_n}$  such that  $F_{n,k_n}^{\nu_n}|_{S_n}$  is a linear isomorphism with  $E_n$ .

For each  $\nu \in \alpha_{\mathbb{C}}^*$ , define

$$d_n(\nu) = \det((F_{n,k_n}^\nu|_{S_n}) \circ (F_{n,k_n}^{\nu_n}|_{S_n})^{-1}),$$

so that  $d_n$  is a nonzero polynomial function on  $\alpha_{\mathbb{C}}^*$ . If  $d_n(\nu) \neq 0$ , then clearly  $L_n^\nu = E_n$ . The theorem now follows by taking  $\mathcal{F}$  to be the set of the  $d_n$  as  $n$  varies and as  $\gamma \in \hat{M}$  varies. Q.E.D.

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